

A blow-up criterion for the compressible liquid crystals system

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Abstract In this paper, we establish a blow-up criterion for the compressible liquid crystals equations in terms of the gradient of the velocity only, similar to the Beale-Kato-Majda criterion [1] for ideal incompressible flows and the criterion obtained by Huang and Xin [8] for the compressible Navier-Stokes equations.

Key words Blow-up criterion; Strong solutions; Liquid crystals equations; Compressible Navier-Stokes equations

2000 MR Subject Classification 76N10, 35M10, 82D30

1 Introduction

In this paper we consider the following simplified model of the Ericksen-Leslie theory for nematic liquid crystals and study a blow-up criterion for it.

$$\rho_t + \operatorname{div}(\rho u) = 0, \quad (1.1)$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p = \mu \Delta u - \lambda \operatorname{div}(\nabla d \odot \nabla d - \frac{1}{2}(|\nabla d|^2 + F(d))I), \quad (1.2)$$

$$d_t + u \cdot \nabla d = \nu(\Delta d - f(d)) \quad (1.3)$$

in $\Omega \times (0, T)$, for a bounded smooth domain Ω in \mathbb{R}^3 .

In the above system, the velocity field $u(x, t)$ of the flow, the direction field $d(x, t)$ representing the orientation parameter of the liquid crystal are vectors in \mathbb{R}^3 . The density $\rho(x, t)$ is a scalar and p is the pressure dependent on the density ρ . μ, λ, ν are positive

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physical constants. The unusual term $\nabla d \odot \nabla d$ denotes the 3×3 matrix whose (i, j) -th element is given by $\sum_{k=1}^3 \partial_{x_i} d_k \partial_{x_j} d_k$ and I is the unite matrix. $f(d)$ is a polynomial of d which satisfies $f(d) = \frac{\partial}{\partial d} F(d)$ where $F(d)$ is the bulk part of the elastic energy. Usually we choose $F(d)$ to be the Ginzburg-Landau penalization, that is, $F(d) = \frac{1}{4\sigma^2}(|d|^2 - 1)^2$ and $f(d) = \frac{1}{\sigma^2}(|d|^2 - 1)d$, where σ is a positive constant.

As the paper [11], we assume the pressure p satisfies

$$p = p(\cdot) \in C^1[0, \infty), \quad p(0) = 0. \quad (1.4)$$

The authors of the paper [11] have proved the following local existence of strong solutions to (1.1)-(1.3) with initial data: $\forall x \in \Omega$,

$$\rho(0, x) = \rho_0 \geq 0, \quad u(0, x) = u_0, \quad d(0, x) = d_0(x), \quad (1.5)$$

boundary conditions: $\forall (t, x) \in (0, T) \times \partial\Omega$,

$$u(t, x) = 0, \quad d(t, x) = d_0(x), \quad |d_0(x)| = 1, \quad (1.6)$$

and some compatibility condition on the initial data:

$$\mu \triangle u_0 - \lambda \operatorname{div}(\nabla d_0 \otimes \nabla d_0 - \frac{1}{2}(|\nabla d_0|^2 + F(d_0))I) - \nabla p_0 = \rho_0^{\frac{1}{2}} g \quad \text{for some } g \in L^2. \quad (1.7)$$

Throughout this paper, we adopt the following simplified notations for Sobolev spaces

$$L^q = L^q(\Omega), \quad W^{k,q} = W^{k,q}(\Omega), \quad H^k = H^k(\Omega), \quad H_0^1 = H_0^1(\Omega).$$

Proposition 1. If (ρ_0, u_0, d_0) satisfies the following regularity condition

$$\rho_0 \in W^{1,6}, \quad u_0 \in H_0^1 \cap H^2 \quad \text{and} \quad d_0 \in H^3, \quad (1.8)$$

and the compatibility condition (1.7), then there exists a small $T^* \in (0, T)$, and a unique strong solution (ρ, u, d) to (1.1)-(1.3) with initial-boundary data (1.5)-(1.6) such that

$$\begin{aligned} \rho &\in C([0, T^*]; W^{1,6}), & \rho_t &\in C([0, T^*]; L^6), \\ u &\in C([0, T^*]; H_0^1 \cap H^2) \cap L^2(0, T^*; W^{2,6}), & u_t &\in L^2(0, T^*; H_0^1), \\ d &\in C([0, T^*]; H^3), & d_t &\in C([0, T^*]; H_0^1) \cap L^2(0, T^*; H^2), \\ d_{tt} &\in L^2(0, T^*; L^2), & \sqrt{\rho} u_t &\in C([0, T^*]; L^2). \end{aligned} \quad (1.9)$$

It is an interesting and natural question whether there is a global strong solution. The paper [12] has proved there is a global weak solution to the compressible liquid crystals system (1.1)-(1.3) where vacuum is allowed initially. And recently the authors of paper [11] have proved the system (1.1)-(1.3) has a global strong solution with small initial data. Since the compressible liquid crystals system (1.1)-(1.3) is coupled by Navier-Stokes equations and liquid crystals equation, it is expected to non-existence of global strong solutions when vacuum regions are present initially. In order to establish a blow-up criterion for the system (1.1)-(1.3), we turn to the Navier-Stokes equations. There are many results concerning blow-up criteria of the incompressible or compressible flow. It is well known that Beal-Kato-Majda established the following blow-up criterion for the incompressible Euler equation in the paper [1]:

$$\lim_{T \rightarrow T^*} \int_0^{T^*} \|\nabla \times u\|_{L^\infty} dt = +\infty.$$

Similarly, Huang and Xin [8] also give a blow-up criterion for the isentropic compressible Navier-Stokes equations as follows:

$$\lim_{T \rightarrow T^*} \int_0^{T^*} \|\nabla u\|_{L^\infty} dt = +\infty$$

Inspired by these ideas, we establish the following criterion for the compressible liquid crystals system (1.1)-(1.3):

Theorem 1. (Blow-up Criterion) Assume that the initial data satisfies the regularity (1.8) and the compatibility condition (1.7). Let (ρ, u, d) be the unique strong solution to the problem (1.1)-(1.3) with the initial boundary conditions (1.5)-(1.6). If T^* is the maximal time of the existence and T^* is finite, then

$$\lim_{T \rightarrow T^*} \int_0^{T^*} \|\nabla u\|_{L^\alpha}^\beta + \|u\|_{W^{1,\infty}} dt = +\infty \quad (1.10)$$

where α, β satisfy

$$\frac{3}{\alpha} + \frac{2}{\beta} < 2 \quad \text{and} \quad \beta \geq 4. \quad (1.11)$$

Remark 1. This criterion given by theorem 1 only involves the velocity u because thanks to the constraint (1.11), the first part of (1.10) plays a role as the direction d .

As usual, we will prove theorem 1 by contradiction in the next section.

2 Proof of Theorem

Let (ρ, u, d) be the unique strong solution to the problem (1.1)-(1.6). We assume the opposite to (1.10) holds, i.e.

$$\lim_{T \rightarrow T^*} \int_0^{T^*} \|\nabla u\|_{L^\alpha}^\beta + \|u\|_{W^{1,\infty}} dt \leq C < +\infty.$$

Hence for all $T < T^*$

$$\int_0^T \|\nabla u\|_{L^\alpha}^\beta + \|u\|_{W^{1,\infty}} dt \leq C, \quad (2.1)$$

from which we will get the same regularity at time T^* as the initial data, a contraction to the maximality of T^* . Thanks to the assumption (1.11) on (α, β) , we have by interpolation

$$\int_0^T \|u\|_{L^\infty}^2 dt, \int_0^T \|\nabla u\|_{L^2}^4 dt, \int_0^T \|\nabla u\|_{L^3}^2 dt \leq C. \quad (2.2)$$

In the following proof, we will employ energy law and higher order energy law.

2.1 Estimate for ρ

It is easy to see that the continuity equation (1.1) on the characteristic curve $\frac{d}{dt}\chi(t) = u(t, \chi(t))$ can be written as

$$\frac{d}{dt}\rho(t, \chi(t)) = -\rho(t, \chi(t))\operatorname{div}u(t, \chi(t)).$$

So

$$\rho(t, \chi(t)) = \rho(0, \chi(0)) \exp\left(-\int_0^t \operatorname{div}u(\tau, \chi(\tau)) d\tau\right) \quad (2.3)$$

Thus

$$0 \leq \rho(t, x) \leq \|\rho_0\|_{L^\infty} \exp\left(\int_0^T \|\operatorname{div}u\|_{L^\infty} dt\right) \leq C \quad \forall (t, x) \in [0, T] \times \overline{\Omega}. \quad (2.4)$$

According to the assumption (1.4) on the pressure p and the above estimate (2.4),

$$\sup_{0 \leq t \leq T} \{\|p(\rho)\|_{L^\infty}, \|p'(\rho)\|_{L^\infty}\} \leq C. \quad (2.5)$$

As the final section of the paper [3], we construct sequences $\{\rho_0^k\}$ and $\{u^k\}$ of smooth scalar and vector fields such that

$$\begin{aligned} \rho_0^k &\in H^2 \cap C^2(\overline{\Omega}), \quad u^k \in L^2(0, T; H_0^1 \cap H^3) \cap C^2([0, T] \times \overline{\Omega}) \quad \text{and} \\ \|\rho_0^k - \rho_0\|_{W^{1,6}} + \int_0^T \|\nabla(u^k - u)(t)\|_{W^{1,6}}^2 dt &\rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (2.6)$$

Then it follows from the classical linear hyperbolic theory that there is a unique solution $\rho^k \in C^2([0, T] \times \overline{\Omega})$ to the following problem:

$$\begin{aligned} \rho_t + \operatorname{div}(\rho u^k) &= 0 \quad \text{in } (0, T) \times \Omega, \\ \rho(0) &= \rho_0^k \quad \text{in } \Omega. \end{aligned} \quad (2.7)$$

The final section of the paper [3] proves that for each fixed $t \in [0, T]$,

$$\rho^k(t) \rightarrow \rho(t) \quad \text{weakly in } W^{1,6}. \quad (2.8)$$

Applying the operator ∇ to the equation (2.7), then multiplying by $\nabla \rho^k$ and integrating over Ω give us we get

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} |\nabla \rho^k|^2 dx \\ &= - \int_{\Omega} |\nabla \rho^k|^2 \operatorname{div} u^k dx - 2 \int_{\Omega} \rho^k \nabla \rho^k \nabla \operatorname{div} u^k dx - 2 \int_{\Omega} (\nabla \rho^k \cdot \nabla u^k) \nabla \rho^k dx \\ &\leq C \|\nabla \rho^k\|_{L^2}^2 \|\nabla u^k\|_{L^\infty} + C \|\nabla \rho^k\|_{L^2} \|\nabla \operatorname{div} u^k\|_{L^2}, \end{aligned}$$

that is,

$$\frac{d}{dt} \|\nabla \rho^k\|_{L^2} \leq C \|\nabla \rho^k\|_{L^2} \|\nabla u^k\|_{L^\infty} + C \|\nabla \operatorname{div} u^k\|_{L^2}$$

Applying Gronwall's inequality to it, we obtain

$$\|\nabla \rho^k\|_{L^2} \leq (\|\rho_0^k\|_{H^1} + C \int_0^t \|\nabla \operatorname{div} u^k\|_{L^2} d\tau) \exp(C \int_0^t \|\nabla u^k\|_{L^\infty} d\tau), \quad \forall t \in [0, T].$$

Hence because of the assumption (2.6) and the convergence (2.8), we can get

$$\|\nabla \rho\|_{L^2} \leq (\|\rho_0\|_{H^1} + C \int_0^t \|\nabla \operatorname{div} u\|_{L^2} d\tau) \exp(C \int_0^t \|\nabla u\|_{L^\infty} d\tau) \quad \forall t \in [0, T]. \quad (2.9)$$

As the above similar process, we obtain

$$\|\nabla \rho\|_{L^6} \leq (\|\rho_0\|_{W^{1,6}} + C \int_0^t \|\nabla \operatorname{div} u\|_{L^6} d\tau) \exp(C \int_0^t \|\nabla u\|_{L^\infty} d\tau) \quad \forall t \in [0, T]. \quad (2.10)$$

2.2 Energy law

Multiplying the momentum equation (1.2) by u and then integrating over Ω , we can obtain

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |u|^2 dx + \int_{\Omega} u \cdot \nabla p dx = -\mu \int_{\Omega} |\nabla u|^2 dx - \lambda \int_{\Omega} (u \cdot \nabla) d \cdot (\Delta d - f(d)) dx. \quad (2.11)$$

Because of the estimate (2.5), we have

$$\left| \int_{\Omega} u \cdot \nabla p dx \right| = \left| \int_{\Omega} p \operatorname{div} u dx \right| \leq \epsilon \int_{\Omega} |\nabla u|^2 dx + C\epsilon^{-1}. \quad (2.12)$$

By liquid crystals equation (1.3), we can get

$$\int_{\Omega} (u \cdot \nabla) d \cdot (\Delta d - f(d)) dx = \frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla d|^2 + F(d) dx + \nu \int_{\Omega} |\Delta d - f(d)|^2 dx. \quad (2.13)$$

So substituting (2.12) and (2.13) into the corresponding terms of (2.11) and taking ϵ small enough give us

$$\frac{dE}{dt} + \int_{\Omega} |\Delta d - f(d)|^2 dx + \int_{\Omega} |\nabla u|^2 dx \leq C \quad (2.14)$$

where

$$E = \int_{\Omega} \rho |u|^2 + |\nabla d|^2 + F(d) dx.$$

Applying Gronwall's inequality to (2.14), we can obtain the desire energy law of the liquid crystals system

$$\sup_{0 \leq t \leq T} \int_{\Omega} \rho |u|^2 + |\nabla d|^2 + F(d) dx + \int_0^T \int_{\Omega} |\Delta d - f(d)|^2 dx dt + \int_0^T \int_{\Omega} |\nabla u|^2 dx dt \leq C. \quad (2.15)$$

2.3 Estimate for d

Multiply the liquid equation (1.3) by d , we know that $|d| \leq 1$ by the maximal principle of parabolic equation. So $f(d)$ and $F(d)$ are bounded.

Lemma 1.

$$\sup_{0 \leq t \leq T} \|d\|_{H^2}^2 + \int_0^T \|\nabla d_t\|_{L^2}^2 dt \leq C. \quad (2.16)$$

Proof. Multiplying (1.3) by Δd_t , we have

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} |\Delta d|^2 dx + \int_{\Omega} |\nabla d_t|^2 dx \\
& \leq C \left(\int_{\Omega} u \cdot \nabla d \Delta d_t dx + \int_{\Omega} (|d|^2 - 1) d \Delta d_t dx \right) \\
& \leq C \left(\int_{\Omega} |\nabla u| |\nabla d| |\nabla d_t| dx + \int_{\Omega} |u| |\nabla^2 d| |\nabla d_t| dx + \int_{\Omega} |\nabla d| |\nabla d_t| dx \right) \\
& \leq \epsilon \|\nabla d_t\|_{L^2}^2 + C\epsilon^{-1} \|\nabla u\|_{L^3}^2 \|\nabla d\|_{L^6}^2 + C\epsilon^{-1} \|\nabla^2 d\|_{L^2}^2 \|u\|_{L^\infty}^2 + C\epsilon^{-1} \|\nabla d\|_{L^2}^2 \\
& \leq \epsilon \|\nabla d_t\|_{L^2}^2 + C\epsilon^{-1} \|\nabla u\|_{L^3}^2 (\|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) + C\epsilon^{-1} \|\nabla^2 d\|_{L^2}^2 \|u\|_{L^\infty}^2 \\
& \quad + C\epsilon^{-1} \|\nabla d\|_{L^2}^2 \\
& \leq \epsilon \|\nabla d_t\|_{L^2}^2 + C\epsilon^{-1} \|\nabla u\|_{L^3}^2 (\|\Delta d\|_{L^2}^2 + \|d_0\|_{H^2}^2 + C) \\
& \quad + C\epsilon^{-1} (\|\Delta d\|_{L^2}^2 + \|d_0\|_{H^2}^2) \|u\|_{L^\infty}^2 + C\epsilon^{-1}
\end{aligned}$$

where in the last inequality we employ the elliptic regularity result $\|\nabla^2 d\|_{L^2} \leq C(\|\Delta d\|_{L^2} + \|d_0\|_{H^2})$ and the energy inequality (2.15).

Taking ϵ small, integrating it over $[0, T]$ and using Gronwall's inequality, we can deduce

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \int_{\Omega} |\Delta d|^2 dx + \int_0^T \int_{\Omega} |\nabla d_t|^2 dx dt \\
& \leq C \left(1 + \int_0^T \|u\|_{L^\infty}^2 + \|\nabla u\|_{L^3}^2 dt \right) \exp \left(\int_0^T \|u\|_{L^\infty}^2 + \|\nabla u\|_{L^3}^2 dt \right) \\
& \leq C
\end{aligned} \tag{2.17}$$

where the last inequality uses the estimate (2.2).

Using the elliptic estimate, (2.17) yields (2.16). ■

Differentiating (1.3) with respect to space gives us

$$\nu \Delta (\nabla d) = \nabla d_t + \nabla (u \cdot \nabla d) + \frac{\nu}{\sigma^2} \nabla [(|d|^2 - 1)d]. \tag{2.18}$$

Applying elliptic regularity result to (2.18), from the estimate (2.16), one can estimate the term $\|\nabla d\|_{H^2}$ as follows

$$\begin{aligned}
\|\nabla d\|_{H^2} & \leq C (\|\nabla d_t\|_{L^2} + \|\nabla (u \cdot \nabla d)\|_{L^2} + \|\frac{\nu}{\sigma^2} \nabla [(|d|^2 - 1)d]\|_{L^2} + \|d_0\|_{H^3} + \|\nabla d\|_{L^2}) \\
& \leq C (\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^3} \|\nabla d\|_{L^6} + \|u\|_{L^\infty} \|\nabla^2 d\|_{L^2} + \|\nabla d\|_{L^2} \|d\|_{L^\infty}^2 \\
& \quad + \|\nabla d\|_{L^2} + \|d_0\|_{H^3}) \\
& \leq C (\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^3} + \|u\|_{L^\infty} + C).
\end{aligned} \tag{2.19}$$

So

$$\int_0^T \|\nabla d\|_{H^2}^2 dt \leq C \int_0^T (\|\nabla d_t\|_{L^2}^2 + \|\nabla u\|_{L^3}^2 + \|u\|_{L^\infty}^2 + C) dt \leq C \quad (2.20)$$

where the second inequality can be obtained by the estimates (2.2) and (2.16).

2.4 Estimate for u

At the beginning, we prove a key lemma

Lemma 2.

$$\sup_{0 \leq t \leq T} \int_{\Omega} \rho |u|^{3+\delta} dx \leq C \quad (2.21)$$

where $\delta(< 1)$ is a small nonnegative constant.

Proof. Multiplying (1.2) by $q|u|^{q-2}u$ and using the estimate (2.5), we can deduce

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \rho |u|^q dx + \int_{\Omega} q |u|^{q-2} (\mu |\nabla u|^2 + \nu \lambda |\Delta d - f(d)|^2 + \mu(q-2) |\nabla |u||^2) dx \\ = & q \int_{\Omega} \operatorname{div}(|u|^{q-2} u) p dx + \lambda q \int_{\Omega} |u|^{q-2} d_t (\Delta d - f(d)) dx \\ \leq & C \int_{\Omega} |u|^{q-2} |\nabla u| dx + \epsilon \int_{\Omega} |u|^{q-2} |\Delta d - f(d)|^2 dx + C\epsilon^{-1} \int_{\Omega} |u|^{q-2} |d_t|^2 dx \\ \leq & \epsilon \int_{\Omega} |u|^{q-2} |\nabla u|^2 dx + C\epsilon^{-1} \int_{\Omega} |u|^{q-2} dx + \epsilon \int_{\Omega} |u|^{q-2} |\Delta d - f(d)|^2 dx \\ & + C\epsilon^{-1} \int_{\Omega} |u|^{q-2} |d_t|^2 dx. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \rho |u|^q dx + \int_{\Omega} |u|^{q-2} (|\nabla u|^2 + |\Delta d - f(d)|^2 + |\nabla |u||^2) dx \\ \leq & C \int_{\Omega} |u|^{q-2} dx + C \int_{\Omega} |u|^{q-2} |d_t|^2 dx. \end{aligned} \quad (2.22)$$

Let $q = 3 + \delta$ and integrate (2.22) over $[0, T]$. Using (2.2), (2.4) and (2.15), we can obtain

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int_{\Omega} \rho |u|^{3+\delta} dx + \int_0^T \int_{\Omega} |u|^{1+\delta} (|\nabla u|^2 + |\Delta d - f(d)|^2 + |\nabla |u||^2) dx dt \\ \leq & C \int_0^T \int_{\Omega} |u|^{1+\delta} dx dt + C \int_0^T \int_{\Omega} |u|^{1+\delta} |d_t|^2 dx dt \\ \leq & C + C \int_0^T \|u\|_{L^6}^4 + \|d_t\|_{L^{\frac{12}{5-\delta}}}^{\frac{8}{3-\delta}} dt \\ \leq & C + C \int_0^T \|d_t\|_{L^3}^4 dt. \end{aligned} \quad (2.23)$$

From the liquid crystal equation (1.3) and using (2.2), (2.16) and (2.20) , we get

$$\begin{aligned}
\int_0^T \|d_t\|_{L^3}^4 dt &\leq \int_0^T \|\Delta d\|_{L^3}^4 dt + \int_0^T \|u \cdot \nabla d\|_{L^3}^4 dt + C \\
&\leq \int_0^T \|\Delta d\|_{L^2}^2 \|\Delta d\|_{L^6}^2 dt + \int_0^T \|\nabla u\|_{L^2}^4 \|\nabla d\|_{L^6}^4 dt + C \\
&\leq C.
\end{aligned} \tag{2.24}$$

Taking (2.24) into (2.23), we obtain the conclusion (2.21) \blacksquare

Lemma 3.

$$\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2) + \int_0^T \|\sqrt{\rho} u_t\|_{L^2}^2 dt \leq C(1 + \eta^{-1}) + \eta \int_0^T \|u_t\|_{L^2}^2 dt \tag{2.25}$$

where σ is a small positive constant and will be determined later.

Proof. Multiplying the momentum equation (1.2) by u_t , integrating over Ω and then using Young's inequality, we have

$$\begin{aligned}
&\frac{\mu}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} \rho |u_t|^2 dx \\
&\leq 2 \int_{\Omega} \rho |u|^2 |\nabla u|^2 dx + \int_{\Omega} p \operatorname{div} u_t dx - \int_{\Omega} (u_t \cdot \nabla) d (\Delta d - f(d)) dx.
\end{aligned} \tag{2.26}$$

Using the continuity equation (1.1) gives us

$$\begin{aligned}
\int_{\Omega} p \operatorname{div} u_t dx &= \frac{d}{dt} \int_{\Omega} p \operatorname{div} u dx - \int_{\Omega} p_t \operatorname{div} u dx \\
&= \frac{d}{dt} \int_{\Omega} p \operatorname{div} u dx + \int_{\Omega} p'(\rho) (\nabla \rho \cdot u + \rho \operatorname{div} u) \operatorname{div} u dx.
\end{aligned} \tag{2.27}$$

Using the liquid crystal equation (1.3) ,we can get

$$\int_{\Omega} (u_t \cdot \nabla) d (\Delta d - f(d)) dx = \frac{1}{\nu} \int_{\Omega} (u_t \cdot \nabla) d (d_t + u \cdot \nabla d) dx. \tag{2.28}$$

Substituting the above equations (2.27) and (2.28) into (2.26), integrating over $(0, t)$ and using Young's inequality, we obtain

$$\begin{aligned}
&\int_{\Omega} |\nabla u|^2 dx + \int_0^t \int_{\Omega} \rho |u_t|^2 dx d\tau \\
&\leq C + C \int_0^t \int_{\Omega} \rho |u|^2 |\nabla u|^2 dx d\tau + C \int_{\Omega} p^2(\rho) dx + C \int_0^t \int_{\Omega} p'(\rho) (\nabla \rho \cdot u \\
&\quad + \rho \operatorname{div} u) \operatorname{div} u dx d\tau + C \int_0^t \int_{\Omega} |u_t| |\nabla d| |\Delta d| + |u_t| |\nabla d| |f(d)| dx d\tau.
\end{aligned} \tag{2.29}$$

In order to estimate the second term of the right side of (2.29), we need to control $\|u\|_{H^2}$. Thanks to the estimate (2.21), we obtain

$$\begin{aligned}
\int_{\Omega} \rho |u|^2 |\nabla u|^2 dx &\leq C \int_{\Omega} \rho^{\frac{2}{3+\delta}} |u|^2 |\nabla u|^2 dx \\
&\leq C \|\rho^{\frac{2}{3+\delta}} |u|^2\|_{L^{\frac{3+\delta}{2}}} \|\nabla u\|_{L^{\frac{3+\delta}{1+\delta}}}^2 \\
&\leq \epsilon^2 \|\nabla u\|_{H^1}^2 + C\epsilon^{-2} \|\nabla u\|_{L^2}^2
\end{aligned} \tag{2.30}$$

where in the last inequality we use the inequality (2.21), the interpolation inequality and Young's inequality.

Rewriting the momentum equation (1.2),

$$\mu \Delta u = \rho u_t + \rho u \cdot \nabla u + \nabla p + \lambda (\nabla d)^T (\Delta d - f(d)).$$

Using elliptic estimate and the inequality (2.30), we can get

$$\begin{aligned}
\|u\|_{H^2} &\leq C(\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + \|\nabla p\|_{L^2} + \|(\nabla d)^T (\Delta d - f(d))\|_{L^2}) \\
&\leq C(\|\sqrt{\rho} u_t\|_{L^2} + \epsilon \|\nabla u\|_{H^1} + \epsilon^{-1} \|\nabla u\|_{L^2} + \|\nabla \rho\|_{L^2} \\
&\quad + \|(\nabla d)^T (d_t + u \cdot \nabla d)\|_{L^2}) \\
&\leq C(\|\sqrt{\rho} u_t\|_{L^2} + \epsilon \|\nabla u\|_{H^1} + \epsilon^{-1} \|\nabla u\|_{L^2} + \|\nabla \rho\|_{L^2} \\
&\quad + \|\nabla d\|_{L^3} \|d_t\|_{L^6} + \|\nabla d\|_{L^6}^2 \|u\|_{L^6}).
\end{aligned}$$

Taking ϵ small enough and using the estimate (2.16), we can get from the above inequality.

$$\|u\|_{H^2} \leq C(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \rho\|_{L^2} + \|\nabla d_t\|_{L^2}). \tag{2.31}$$

We continue our proof.

Thanks to (2.31),

$$\int_0^t \int_{\Omega} \rho |u|^2 |\nabla u|^2 dx d\tau \tag{2.32}$$

$$\leq \int_0^t (\epsilon^2 \|\nabla u\|_{H^1}^2 + C\epsilon^{-2} \|\nabla u\|_{L^2}^2) d\tau \tag{2.33}$$

$$\begin{aligned}
&\leq C\epsilon^2 \int_0^t (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) d\tau + C\epsilon^{-2} \int_0^t \|\nabla u\|_{L^2}^2 d\tau \\
&\leq C\epsilon^2 \int_0^t \|\sqrt{\rho} u_t\|_{L^2}^2 d\tau + C(\epsilon^2 + \epsilon^{-2}) \int_0^t \|\nabla u\|_{L^2}^2 d\tau + C\epsilon^2 \int_0^t \|\nabla \rho\|_{L^2}^2 d\tau + C\epsilon^2.
\end{aligned} \tag{2.34}$$

where the last inequality utilizes the estimate (2.20).

Using the above estimates (2.2), (2.4), (2.5), (2.16) and (2.20), we can obtain

$$\int_0^t \int_{\Omega} p'(\rho)(\nabla \rho \cdot u) |\operatorname{div} u| dx d\tau \leq C \int_0^t \|\nabla \rho\|_{L^2}^2 \|u\|_{L^\infty} d\tau + C \int_0^t \|\nabla u\|_{L^2}^2 \|u\|_{L^\infty} d\tau \quad (2.35)$$

$$\int_0^t \int_{\Omega} p'(\rho) \rho |\operatorname{div} u|^2 dx d\tau \leq C \int_0^t \|\nabla u\|_{L^2}^2 d\tau \leq C, \quad (2.36)$$

$$\begin{aligned} \int_0^t \int_{\Omega} |u_t| |\nabla d| \triangle d dx d\tau &\leq \int_0^t \|u_t\|_{L^2} \|\nabla d\|_{L^3} \|\triangle d\|_{L^6} d\tau \\ &\leq \eta \int_0^t \|u_t\|_{L^2}^2 d\tau + C\eta^{-1} \int_0^t \|d\|_{H^3}^2 d\tau \\ &\leq \eta \int_0^t \|u_t\|_{L^2}^2 d\tau + C\eta^{-1} \end{aligned} \quad (2.37)$$

and

$$\begin{aligned} \int_0^t \int_{\Omega} |u_t| |\nabla d| |f(d)| dx d\tau &\leq C \int_0^t \|u_t\|_{L^2} \|\nabla d\|_{L^2} (\|d\|_{L^\infty}^2 + 1) \|d\|_{L^\infty} d\tau \\ &\leq \eta \int_0^t \|u_t\|_{L^2}^2 d\tau + C\eta^{-1}. \end{aligned} \quad (2.38)$$

Substituting (2.5) and (2.34)-(2.38) into (2.29) and taking ϵ small, we can obtain

$$\begin{aligned} &\int_{\Omega} |\nabla u|^2 dx + \int_0^t \int_{\Omega} \rho |u_t|^2 dx d\tau \\ &\leq C(1 + \eta^{-1}) + C \int_0^t (\|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2)(1 + \|u\|_{L^\infty}) d\tau + \eta \int_0^t \|u_t\|_{L^2}^2 d\tau. \end{aligned} \quad (2.39)$$

Taking (2.31) into (2.9), we obtain

$$\zeta \|\nabla \rho\|_{L^2}^2 \leq C\zeta(1 + \int_0^t \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 d\tau) \quad (2.40)$$

Taking ζ small, combining (2.39) with (2.40) and using the estimates (2.2) and (2.16), we have

$$\begin{aligned} &\int_{\Omega} (|\nabla u|^2 + |\nabla \rho|^2) dx + \int_0^t \int_{\Omega} \rho |u_t|^2 dx d\tau \\ &\leq C(1 + \eta^{-1}) + C \int_0^t (\|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2)(\|\nabla u\|_{L^\infty} + 1) d\tau + \eta \int_0^t \|u_t\|_{L^2}^2 d\tau. \end{aligned} \quad (2.41)$$

Applying generalized Gronwall's inequality to (2.41), we deduce

$$\sup_{0 \leq t \leq T} \int_{\Omega} (|\nabla u|^2 + |\nabla \rho|^2) dx + \int_0^T \int_{\Omega} \rho |u_t|^2 dx d\tau \leq C(1 + \eta^{-1}) + \eta \int_0^T \|u_t\|_{L^2}^2 d\tau. \quad \blacksquare$$

2.5 Higher order energy inequality

We will use higher order energy inequality to deduce the following lemma:

Lemma 4.

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \\ & + \int_0^T (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|(\Delta d - f(d))_t\|_{L^2}^2) dt \leq C. \end{aligned} \quad (2.42)$$

Proof. Rewrite the momentum equation (1.2) in a non conservative form as

$$\rho u_t + \rho u \cdot \nabla u + \nabla p_t = \mu \Delta u - \lambda (\nabla d)^T (\Delta d - f(d)). \quad (2.43)$$

Then differentiate the above equation (2.43) with respect to time, multiply the resulting equation by u_t and integrate it over Ω to get

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |u_t|^2 dx + \int_{\Omega} \mu |\nabla u_t|^2 dx - \int_{\Omega} p_t \operatorname{div} u_t dx \\ = & - \int_{\Omega} \rho u \cdot \nabla \left(\frac{1}{2} |u_t|^2 \right) + (u \cdot \nabla u) \cdot u_t + \rho (u_t \cdot \nabla u) \cdot u_t dx \\ & - \lambda \int_{\Omega} (u_t \cdot \nabla) d_t \cdot (\Delta d - f(d)) + (u_t \cdot \nabla) d \cdot (\Delta d - f(d))_t dx. \end{aligned} \quad (2.44)$$

Differentiating liquid crystals equation (1.3) with respect to time derives

$$u_t \cdot \nabla d = \nu (\Delta d - f(d))_t - d_{tt} - u \cdot \nabla d_t.$$

Then

$$\begin{aligned} & \int_{\Omega} (u_t \cdot \nabla) d \cdot (\Delta d - f(d))_t dx \\ = & \int_{\Omega} |\nu (\Delta d - f(d))_t|^2 - d_{tt} \Delta d_t + d_{tt} f(d)_t - (u \cdot \nabla) d_t (\Delta d - f(d))_t dx \\ = & \int_{\Omega} -(u_t \cdot \nabla d + u \cdot \nabla d_t) f(d)_t + (\nu (f(d))_t - u \cdot \nabla d_t) (\Delta d - f(d))_t dx \\ & + \int_{\Omega} |\nu (\Delta d - f(d))_t|^2 dx + \frac{d}{dt} \int_{\Omega} |\nabla d_t|^2 dx. \end{aligned} \quad (2.45)$$

By the continuity (1.1), the term of p in (2.44) becomes

$$\int_{\Omega} p_t \operatorname{div} u_t dx = - \int_{\Omega} p'(\rho) (\nabla \rho \cdot u + \rho \operatorname{div} u) \operatorname{div} u_t dx. \quad (2.46)$$

Substituting (2.45) and (2.46) into (2.44), we get the first order energy inequality

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \rho |u_t|^2 + \lambda |\nabla d_t|^2 \right) dx + \int_{\Omega} \mu |\nabla u_t|^2 + \lambda \nu^2 |(\Delta d - f(d))_t|^2 dx \\
\leq & C \left(\int_{\Omega} \rho |u| |\nabla u_t| |u_t| + \rho |u| |u_t| |\nabla u|^2 + \rho |u|^2 |u_t| |\nabla^2 u| + \rho |u|^2 |\nabla u| |\nabla u_t| dx \right. \\
& + \int_{\Omega} \rho |u_t|^2 |\nabla u| dx + \int_{\Omega} |(u_t \cdot \nabla d) f(d)_t| + |(u \cdot \nabla d_t) f(d)_t| dx \\
& + \int_{\Omega} |(\Delta d - f(d))_t f(d)_t| + |(u \cdot \nabla d_t)(\Delta d - f(d))_t| dx \\
& + \int_{\Omega} |(u_t \cdot \nabla) d_t \cdot (\Delta d - f(d))| dx + \int_{\Omega} |p'(\rho)| |\nabla \rho| |u| |\operatorname{div} u_t| dx \\
& \left. + \int_{\Omega} \rho |p'(\rho)| |\operatorname{div} u| |\operatorname{div} u_t| dx \right) \\
= & C \sum_{i=1}^{12} I_i. \tag{2.47}
\end{aligned}$$

Now we estimate each term I_i . In the following calculations, we will make full use of Sobolev inequality, Hölder inequality and estimate (2.4), (2.16), (2.5) and (2.31).

$$\begin{aligned}
I_1 & \leq \|\rho\|_{L^\infty}^{\frac{1}{2}} \|u\|_{L^\infty} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u_t\|_{L^2} \\
& \leq \epsilon \|\nabla u_t\|_{L^2}^2 + C \epsilon^{-1} \|u\|_{L^\infty}^2 \|\sqrt{\rho} u_t\|_{L^2}^2, \\
I_2 & \leq C \|u\|_{L^6} \|u_t\|_{L^6} \|\nabla u\|_{L^2} \|\nabla u\|_{L^6} \\
& \leq \epsilon \|u_t\|_{H^1}^2 + C \epsilon^{-1} \|u\|_{H^1}^4 \|u\|_{H^2}^2 \\
& \leq \epsilon \|u_t\|_{H^1}^2 + C \epsilon^{-1} \|u\|_{H^1}^4 (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2), \\
I_3 & \leq C \|u_t\|_{L^6} \|u\|_{L^6}^2 \|\nabla^2 u\|_{L^2} \\
& \leq \epsilon \|u_t\|_{H^1}^2 + C \epsilon^{-1} \|u\|_{H^1}^4 \|u\|_{H^2}^2 \\
& \leq \epsilon \|u_t\|_{H^1}^2 + C \epsilon^{-1} \|u\|_{H^1}^4 (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2), \\
I_4 & \leq C \|u\|_{L^6}^2 \|\nabla u\|_{L^6} \|\nabla u_t\|_{L^2} \\
& \leq \epsilon \|u_t\|_{H^1}^2 + C \epsilon^{-1} \|u\|_{H^1}^4 \|u\|_{H^2}^2 \\
& \leq \epsilon \|u_t\|_{H^1}^2 + C \epsilon^{-1} \|u\|_{H^1}^4 (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2), \\
I_5 & \leq \|\rho\|_{L^\infty}^{\frac{1}{2}} \|\sqrt{\rho} u_t\|_{L^2}^2 \|\nabla u\|_{L^\infty},
\end{aligned}$$

$$\begin{aligned}
I_6 &\leq \|u_t\|_{L^2} \|\nabla d\|_{L^3} (\|d\|_{L^\infty}^2 + 1) \|d_t\|_{L^6} \\
&\leq \epsilon \|u_t\|_{L^2}^2 + C\epsilon^{-1} \|\nabla d_t\|_{L^2}^2, \\
I_7 &\leq C \|u\|_{L^\infty} \|\nabla d_t\|_{L^2} \|d_t\|_{L^2} (\|d\|_{L^\infty}^2 + 1) \\
&\leq C \|u\|_{L^\infty} \|\nabla d_t\|_{L^2}^2, \\
I_8 &\leq \epsilon \|(\Delta d - f(d))_t\|_{L^2}^2 + C\epsilon^{-1} \|f(d)_t\|_{L^2}^2 \\
&\leq \epsilon \|(\Delta d - f(d))_t\|_{L^2}^2 + C\epsilon^{-1} \|d_t\|_{L^2}^2, \\
I_9 &\leq \epsilon \|(\Delta d - f(d))_t\|_{L^2}^2 + C\epsilon^{-1} \|u\|_{L^\infty}^2 \|\nabla d_t\|_{L^2}^2, \\
I_{10} &\leq \|u_t\|_{L^3} \|\nabla d_t\|_{L^2} \|\Delta d\|_{L^6} + \|u_t\|_{L^2} \|\nabla d_t\|_{L^2} (\|d\|_{L^\infty}^2 + 1) \|d\|_{L^\infty} \\
&\leq \epsilon \|\nabla u_t\|_{L^2}^2 + C\epsilon^{-1} \|\nabla d_t\|_{L^2}^2 (1 + \|\Delta d\|_{L^6}^2), \\
I_{11} &\leq \epsilon \|\nabla u_t\|_{L^2}^2 + C\epsilon^{-1} \|\nabla \rho\|_{L^2}^2 \|u\|_{L^\infty}^2
\end{aligned}$$

and

$$I_{12} \leq \epsilon \|\nabla u_t\|_{L^2}^2 + C\epsilon^{-1} \|\nabla u\|_{L^2}^2.$$

Substituting all the estimates into (2.47) and taking ϵ small, we obtain

$$\begin{aligned}
&\frac{d}{dt} \int_{\Omega} \rho |u_t|^2 + |\nabla d_t|^2 dx + \int_{\Omega} |\nabla u_t|^2 + |(\Delta d - f(d))_t|^2 dx \\
&\leq C (\|\sqrt{\rho} u_t\|_{L^2}^2 A(t) + \|\nabla u\|_{L^2}^2 B(t) + \|\nabla \rho\|_{L^2}^2 C(t) + \|\nabla d_t\|_{L^2}^2 D(t)) \quad (2.48)
\end{aligned}$$

where

$$\begin{aligned}
A(t) &= \|u\|_{L^\infty}^2 + \|u\|_{H^1}^4 + \|\nabla u\|_{L^\infty}, \\
B(t) &= \|u\|_{H^1}^4 + 1, \\
C(t) &= \|u\|_{H^1}^4 + \|u\|_{L^\infty}^2, \\
D(t) &= \|u\|_{H^1}^4 + \|u\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|\Delta d\|_{L^6}^2 + 1.
\end{aligned}$$

The estimates (2.2) and (2.20) yield

$$\int_0^T (A(t) + B(t) + C(t) + D(t)) dt \leq C. \quad (2.49)$$

Applying the Gronwall's inequality to the inequality (2.48), we deduce

$$\begin{aligned}
&\sup_{0 \leq t \leq T} (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + \int_0^T \int_{\Omega} |\nabla u_t|^2 + |(\Delta d - f(d))_t|^2 dx d\tau \\
&\leq C \int_0^T (\|\nabla u\|_{L^2}^2 B(t) + \|\nabla \rho\|_{L^2}^2 C(t)) dt \exp \int_0^T (A(t) + D(t)) dt \\
&\leq C \int_0^T (\|\nabla u\|_{L^2}^2 B(t) + \|\nabla \rho\|_{L^2}^2 C(t)) dt. \quad (2.50)
\end{aligned}$$

Combing with the estimate (2.25) and taking η small, we get the desired finial inequality

$$\begin{aligned}
& \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \\
& + \int_0^T (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|(\Delta d - f(d))_t\|_{L^2}^2) dt \\
& \leq C + C \int_0^T (\|\nabla u\|_{L^2}^2 B(t) + \|\nabla \rho\|_{L^2}^2 \mathcal{C}(t)) dt.
\end{aligned} \tag{2.51}$$

Applying Gronwall's inequality to the inequality (2.51) again, we deduce

$$\begin{aligned}
& \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \\
& + \int_0^T (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|(\Delta d - f(d))_t\|_{L^2}^2) dt \\
& \leq C.
\end{aligned} \quad \blacksquare$$

2.6 Estimate for $\|u\|_{H^2}$, $\|d\|_{H^3}$ and $\|\rho\|_{W^{1,6}}$

From the estimate (2.42), (2.31) yields

$$\|u\|_{H^2} \leq C(\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \rho\|_{L^2} + \|\nabla d_t\|_{L^2}) \leq C. \tag{2.52}$$

From the estimate (2.42) and the above inequality (2.52), (2.19) yields

$$\begin{aligned}
\|\nabla d\|_{H^2} & \leq C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^3} + \|u\|_{L^\infty} + C) \\
& \leq C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}} + \|u\|_{H^2} + C) \\
& \leq C.
\end{aligned} \tag{2.53}$$

Lemma 5.

$$\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^6} + \int_0^T \|\nabla^2 u\|_{L^6}^2 dt \leq C. \tag{2.54}$$

Proof. Using the elliptic regularity result $\|\nabla^2 u\|_{L^6} \leq C\|\Delta u\|_{L^6}$ and the above estimates (2.52) and (2.53) give

$$\begin{aligned}
\|\nabla^2 u\|_{L^6} & \leq C(\|\rho u_t\|_{L^6} + \|\rho u \cdot \nabla u\|_{L^6} + \|\nabla p\|_{L^6} + \|(\nabla d)^T(\Delta d - f(d))\|_{L^6}) \\
& \leq C(\|\nabla u_t\|_{L^2} + \|u\|_{L^\infty} \|\nabla u\|_{L^6} + \|\nabla \rho\|_{L^6} + \|\nabla d\|_{L^\infty} \|\Delta d\|_{L^6} \\
& \quad + \|\nabla d\|_{L^6} \|f(d)\|_{L^\infty}) \\
& \leq C(\|\nabla u_t\|_{L^2} + \|u\|_{H^2}^2 + \|\nabla \rho\|_{L^6} + \|d\|_{H^3}^2 + \|d\|_{H^2}) \\
& \leq C(\|\nabla u_t\|_{L^2} + \|\nabla \rho\|_{L^6} + 1).
\end{aligned} \tag{2.55}$$

Taking the above inequality (2.55) into (2.10), we get

$$\|\nabla \rho\|_{L^6} \leq (\|\rho_0\|_{W^{1,6}} + C \int_0^t (\|\nabla u_t\|_{L^2} + \|\nabla \rho\|_{L^6} + 1) d\tau) \exp(C \int_0^t \|\nabla u\|_{L^\infty} d\tau) \quad (2.56)$$

Using the assumption (2.1) and the estimate (2.42), and then applying Gronwall's inequality to (2.56), we obtain

$$\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^6} \leq C. \quad (2.57)$$

Moreover from (2.55) and (2.57) we have

$$\int_0^T \|\nabla^2 u\|_{L^6}^2 dt \leq C. \quad \blacksquare$$

From the Proposition 1, $\|u\|_{H^2}(t)$, $\|\rho\|_{W^{1,6}}(t)$, $\|d\|_{H^3}(t)$, $\|\sqrt{\rho}u_t\|_{L^2}(t)$ are all continuous on time $[0, T^*)$. From the above estimates (2.4), (2.16), (2.42) and (2.52)-(2.54), we see that

$$\begin{aligned} & (\|\rho\|_{W^{1,6}}, \|u\|_{H^2}, \|d\|_{H^3}, \|\sqrt{\rho}u_t\|_{L^2})|_{t=T^*} \\ &= \lim_{t \rightarrow T^*} (\|\rho\|_{W^{1,6}}, \|u\|_{H^2}, \|d\|_{H^3}, \|\sqrt{\rho}u_t\|_{L^2}) \\ &\leq C < \infty. \end{aligned} \quad (2.58)$$

The finite of $\|\sqrt{\rho}u_t\|_{L^2}|_{t=T^*}$ means there is a compatibility condition at time T^* . Hence we can take $(\rho, u, d)|_{t=T^*}$ as the initial data and apply the Proposition 1 to extend our local solution beyond T^* in time which contradicts with the maximality of T^* . Therefore the assumption (2.1) does't hold, that is, (1.10) holds if T^* is the maximal time of the existence and T^* is finite.

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